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# A Control-Theoretic Formulation of the Bunch Train Cavity Interaction

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**Abstract**

The bunch train cavity interaction is an accelerator physics problem, for which a system-theoretic model is lacking. Modal analysis has been used to characterize the system dynamics, exploiting the system's symmetry. Correspondingly, control design has been done using classical frequency-domain-based control. Several shortcomings of these methods are highlighted, all of which are remedied by a new time-domain, system-theoretic model presented herein. The new formulation is a periodic, discrete-time system, amenable to state-space control-design methods.

## 1 Introduction

This paper is concerned with the dynamics and control of a circulating particle beam. We refer to it as the bunch train cavity interaction (BTCI)[1], and though ostensibly a control-theoretic problem, theory and practical design have been largely the work of physicists, who have favored modal analysis (MA). State-space methods have been ignored, and all of the existing feedback designs are exclusively via classical methods and ad hoc extensions. It is not uncommon for current systems to be controlled by several nested loops. This paper devlops a general state-space model and demonstrates its utility in system analysis and control design. The next two paragraphs qualitatively describe the bunch train cavity interaction problem; Figure 1 should serve as an aid.

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[1]known by physicists as the coupled bunch instability problem

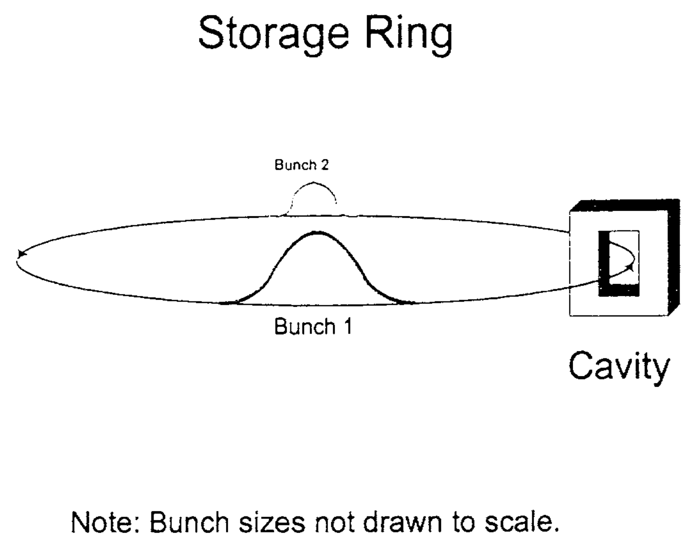


Figure 1: Bunches of charged particles encircle the storage ring and interact with the cavity.

The circulating particle beam is comprised of bunches of particles (usually electrons or protons) of some nominal energy, each separated from the others by distances that are several orders of magnitude larger than the (average) bunch size. The bunches circulate along a nominal orbit, during which they lose energy. This energy loss is compensated by a passive, high-$Q$ resonant cavity magnet (or a group of them), which imparts energy to each bunch as the bunch passes through it. The cavity is powered by an external generator that supplies radio-frequency (rf) power at some nominal level. There exists an equilibrium at which the nominal rf power makes up for each bunch's energy loss exactly, such that the bunches circulate about their orbit forever.

However, perturbations to this system equlibrium may grow to instability, or the damping rate may be too slow. The mechanism is the coupling from the bunches to the cavity (or any other significant source of impedance):

the bunch train acts as a (second) power input to the cavity (aside from the generator), and thus each bunch indirectly affects its fellows. Feedback systems are often used to provide better system behavior. The control law may incorporate information regarding the bunches' energy (or a functionally related quantity) and/or information from the cavity. A typical control lever is via the generator, i.e., modulating the rf incident to the cavity [1]. A further control lever (often used in parallel with the first) is obtained by inserting a broadband kicker magnet in the nominal beam orbit, that, like the cavity, imparts energy to each passing bunch. But unlike the cavity, this actuator contributes a small, strictly corrective energy kick to *each* bunch [2], [3]. Of the two control problems, the former is far more challenging, as the plant consists of the entire bunch train with the cavity, while in the latter, each bunch is treated as a separate plant.

In the following sections of this paper a mathematical description of the bunch train cavity interaction is given (in section 2) that includes a brief coverage of modal analysis. A state-space, system-theoretic plant and control system model is then derived (in section 3). Simulation results are presented (in section 4) for a low-order case.

## 2 Mathematical Formulation

In this section we present all of the necessary subsystem models of the BTCI. With just the information presented here our new formulation can be derived, as is done in section 3.

The simplest component of the overall system is the cavity. Although spatially distributed, it is well modeled for the topic at hand by a lumped transfer function [4]:

$$Z(s) = \frac{R\omega_{\mathrm{r}} s/Q}{s^2 + \omega_{\mathrm{r}} s/Q + \omega_{\mathrm{r}}^2}, \tag{1}$$

where $\omega_{\mathrm{r}}$ is the center frequency (on the order of rf), $Q \gg 1$ is the dimensionless quality factor, and $R$ is the resistance in Ohms. As mentioned in section 1, there are two inputs to $Z(s)$: the supplied rf power expressed in current (Amps) and the bunch train itself. For the moment we assume that the former is constant.

The bunch train is modeled as a (quasi-) periodic Dirac-impulse train. [Each bunch is Gaussian shaped, but given the speed with which it passes through the cavity (on the order of $c$), and moreover, with respect to $Z(s)$, it can be treated as a Dirac impulse $\delta(\cdot)$.] Under a no-perturbation, strictly-periodic condition, each bunch $n = 1, \ldots, N$ of charge $q_n$ passes through the cavity with a constant period $T$ phased some $T_n$ seconds after the first bunch ($T_1 := 0$). That is,

$$i(t) = -\sum_{n=1}^{N} q_n \sum_{p} \delta\left(t - pT - T_n\right) \qquad \text{under no perturbations.} \tag{2}$$

In reality, however, the bunches arrive early or late by no more than $\hat{\tau} \ll \min_{n>1} T_n$ seconds, i.e., the impulses fire *quasi-periodically.* Let $\tau_n[p]$ denote the $n$th bunch's $p$th arrival time deviation from nominal (jitter); hence the true bunch train current is given by

$$i(t) = -\sum_{n=1}^{h} q_n \sum_{p} \delta\left(t - pT - T_n - \tau_n[p]\right). \tag{3}$$

A convenient shorthand is to denote the argument of the above delta function as

$$t - t_{n,p} := t - pT + T_n + \tau_n[p], \tag{4}$$

and for convenience in indexing we allow $t_{0,p} = pT - T_N + \tau_N[p-1]$ (periodicity).

The BTCI system is interesting from a system theory point of view because the jitter is itself driven by perturbations in the cavity's voltage. This is an inherent feedback mechanism whereby cavity voltage perturbations drive and are driven by bunch phase perturbations. More precisely, for some convenient $\omega_c$ in the bandwidth of $Z(j\omega)$, let $\theta_n[p] := \omega_c \tau_n[p]$ denote the $n$th bunch's radian phase deviation on the $p$th arrival, and let $\hat{\theta} := \max_{n,p} |\theta_n[p]|$. Similarly let $\epsilon_n[p]$ denote the bunch's energy deviation (in volts). Defining $\mathbf{x}_n[p] = [\theta_n[p], \epsilon_n[p]]^{\mathrm{T}}$, the bunch's phase-energy dynamics are governed (to first order) by the following difference equation $\forall n$

$$\mathbf{x}_n[p+1] = \mathbf{A}_{\mathrm{b}} \mathbf{x}_n[p] + \mathbf{B}_{\mathrm{b}} v(t_{n,p}),$$

where $v(t_{n,p})$ is the response of $Z(s)$ to $i(t)$ evaluated at time $t = t_{n,p}$. A fundamental result in BTCI theory [4] is that we can write

$$\mathbf{x}_n[p+1] = \mathbf{A}_{\mathrm{b}} \mathbf{x}_n[p] + \mathbf{B}_{\mathrm{b}} v(t_{n-1,p}^{-}) + \mathbf{b}_{\mathrm{b}o}, \tag{5}$$

where $t_{n,p}^{-}$ denotes the time $\eta > 0$ seconds *infinitesimally before* the $n$th bunch's $p$th arrival [see Eq. (4)]. This result says that a bunch's self-induced perturbation as it interacts with the cavity is constant; the effects of previous (bunches') interactions are reflected in $v(t_{n-1,p}^{-})$.

To summarize, the BCTI consists of the continuous time, cavity-dynamics model Eq. (1), and $N$ discrete time, bunch-dynamics models Eq. (5). Each $n$th bunch model interacts with the cavity at times $t_{n,p}$, *which themselves are state dependent*, specifically $\theta_n[p]$ dependent. Thus, although the constituent subsystems are linear, the overall BTCI is nonlinear (of order infinity) owing to the fact that the bunch phases appear as functional time arguments.

## 2.1 Modal Analysis

We sketch here the modal analysis approach *as traditionally applied to the BTCI problem.* Our focus is on control-theoretic aspects, namely, that modal analysis can lead to a nonminimal realization, and that frequency-domain-based control imposes actuator bandwidth constraints. Note that it is possible to obtain the system eigenmodes also using our state-space model developed below.

The discrete-time bunch Eq. (5) is approximated as a continuous-time differential equation $n = 1, \ldots, N$,

$$\ddot{\theta}_n(t) + a_1\dot{\theta}_n(t) + a_0\theta_n(t) = b_1 v_n(t), \tag{6}$$

where all coefficients arise from the matrices of Eq. (5), and $v_n(t)$ represents the voltage seen by the $n$th bunch. The latter is a function of all $N$ bunches' phases $\theta_1, \ldots, \theta_N$ up until time $t$. Specifically,

$$v_n(t) = \sum_{m=1}^{N} \sum_{p} z\left(pT + T_n - T_m + \tau_n(t) - \tau_m(t - pT + T_n - T_m)\right), \tag{7}$$

with $\tau_n(t) = \theta_n(t)/\omega_c$. The first-order solution of Eq. (6) is obtained by taking its Fourier transform and keeping only the first two moments with respect to the $\tau$ arguments.

This approach is tractable under the following symmetry assumption: $S_n := T_n - T_{n-1} = T/N, \forall n$, i.e., the nominal distance between bunches is fixed. To force this condition to hold, one must append phantom bunches to the true system, viz., bunches of zero charge as proposed in [5]. For example, the $N = 2$, $q_1 = q_2 = q$ case, in which $T_2 = T/3$ ($T_1 = 0$), is modeled under modal analysis as an $N = 3$ case in which $T_3 = 2T/3$ and $q_3 = 0$. Thus *modal analysis of an asymmetric case leads to a nonminimal realization*, as more bunch models [Eqs. (6) or (5)] have to be appended to the overall system.

The quantity of interest in this modal analysis is the voltage-induced growth rate, which for mode $l, l = 0, \ldots, N-1$ is proportional to

$$\sum_{k=-\infty}^{\infty} (k\omega_c + l\omega_o + \sqrt{a_0}) Z(k\omega_c + l\omega_o + \sqrt{a_0}),$$

with $\omega_o = 2\pi/T$. Thus one can damp a given mode by feeding back the cavity voltage signal through the generator, using a comb filter tuned to $\omega_c + l\omega_o + \sqrt{a_0}$ as a controller [1]. However, the viability of such a feedback scheme is limited to the bandwidth of the generator. E.g., it is not possible to damp mode $l = 1$ if the generator bandwidth is on the order of $\omega_c + l\omega_o$.

## 3 A State-Space Formulation

Our state-space, time-domain formulation has been inspired by the sampling jitter problem [6], except that in the present case the jitter *is itself a state.* We now obtain a linear model as a first-order approximation to the full system. We also make use of the Floquet Theorem [7], under which a linear periodic system's eigenvalues are determined by its behavior over one period only. This means that to evaluate growth rates (eigenvalues) we need only evaluate the eigenvaules of the one-period system matrix.

Describe the cavity in an equivalent state-space realization $(\mathbf{A}_c, \mathbf{B}_c, \mathbf{C}_c)$, with state vector $\mathbf{x}_c(t)$, input $i(t)$, and ouput $v(t)$. Thus

$$\begin{aligned} \dot{\mathbf{x}}_c(t) &= \mathbf{A}_c \mathbf{x}_c(t) + \mathbf{B}_c i(t) \\ v(t) &= \mathbf{C}_c \mathbf{x}_c(t). \end{aligned}$$

Thus (after dropping the constant $\mathbf{b}_{bo}$) Eq. (5) is rewritten as

$$\mathbf{x}_n[p+1] = \mathbf{A}_b \mathbf{x}_n[p] + \mathbf{B}_b \mathbf{C}_c \mathbf{x}_c(t^-_{n-1,p}). \tag{8}$$

We seek to evaluate the BTCI at *fixed time instants*, specifically at times $pT + T_n$, and not at the (variable) interaction times $t_{n,p}$.

As in section 2, for convenience in indexing, we let $t_{0,p} := t_{N,p-1}$, $q_0 := q_N$, and $T_0 := T - T_N$. Recall that we have also defined the the bunch spacing $S_n := T_n - T_{n-1}$. Thus for each $n$th bunch

$$\begin{aligned} \mathbf{x}_c(t^-_{n,p}) &= e^{(t^-_{n,p} - t^-_{n-1,p})\mathbf{A}_c} \mathbf{x}_c(t^-_{n-1,p}) + \int_{t^-_{n-1,p}}^{t^-_{n,p}} e^{(t^-_{n,p} - t)\mathbf{A}_c} \mathbf{B}_c i(t) \mathrm{d}t \\ &= e^{(t^-_{n,p} - t^-_{n-1,p})\mathbf{A}_c} \mathbf{x}_c(t^-_{n-1,p}) - e^{(t^-_{n,p} - t^-_{n-1,p} - \eta)\mathbf{A}_c} \mathbf{B}_c q_{n-1}. \end{aligned} \tag{9}$$

Using the invertible map $e^{t\mathbf{A}_c}$, define

$$\mathbf{x}_c(t_{n,p}^-) =: e^{(\tau_n[p]-\eta)\mathbf{A}_c}\bar{\mathbf{x}}_c(pT + T_n) \tag{10}$$

as the cavity state just before the $n$th bunch's $p$th arrival *propogated to the convenient fixed time* $pT + T_n$. Restating Eq. (9) we have

$$\bar{\mathbf{x}}_c(pT + T_n) = e^{S_n\mathbf{A}_c}\bar{\mathbf{x}}_c(pT + T_{n-1}) - q_{n-1}e^{(S_n - \tau_{n-1}[p]-\eta)\mathbf{A}_c}\mathbf{B}_c. \tag{11}$$

The bunch model of Eq. (5) is similarly rewritten as

$$\mathbf{x}_n[p+1] = \mathbf{A}_b\mathbf{x}_n[p] + \mathbf{B}_b\mathbf{C}_c e^{(S_n+\tau_n[p]-\eta)\mathbf{A}_c}\bar{\mathbf{x}}_c(pT + T_{n-1}). \tag{12}$$

Taken together, the rewritten cavity and bunch subsystems [Eqs. (11) and 12, respectively] are nonlinear in the state $\theta_n/\omega_c$. They can be linearized by taking the first two terms in the matrix exponential (when this is valid). However, given that $Q \gg 1$, i.e., that the cavity is a bandlimited resonator, and that we are concerned with a small signal model, i.e., $\hat{\tau} \ll 1$, a better procedure is to cast the cavity system in terms of the in-phase and quadrature (I and Q) responses [8] about a "carrier" radian frequency $\omega_c$. We impose the easily satisfied conditions that $\omega_c$ falls in the bandwidth of the cavity and that $\omega_c T_n \bmod 2\pi = 0, \forall n$. This method yields a baseband BTCI model as developed in the next three paragraphs.

The inverse transform $z(t)$ of Eq. (1) can be written as a sum of orthogonal impulse responses, which in turn can be written in state-space equivalents:

$$\begin{aligned}
z(t) &= z_I(t)\cos\omega_c t - z_Q(t)\sin\omega_c t \\
&= \mathbf{C}_I e^{t\mathbf{A}_I}\mathbf{B}_I\cos\omega_c t - \mathbf{C}_Q e^{\mathbf{A}_Q}\mathbf{B}_Q\sin\omega_c t \\
&= \begin{bmatrix} \mathbf{C}_I & \mathbf{C}_Q \end{bmatrix}\begin{bmatrix} e^{t\mathbf{A}_I}\cos\omega_c t & \\ & -e^{t\mathbf{A}_Q}\sin\omega_c t \end{bmatrix}\begin{bmatrix} \mathbf{B}_I \\ \mathbf{B}_Q \end{bmatrix}.
\end{aligned} \tag{13}$$

The in-phase and quadrature state-space models $(\mathbf{A}_I, \mathbf{B}_I, \mathbf{C}_I)$ and $(\mathbf{A}_Q, \mathbf{B}_Q, \mathbf{C}_Q)$ have states $\mathbf{x}_I$ and $\mathbf{x}_Q$, respectively.

Our assumptions and conditions yield:

$$e^{(S_n-\tau_n[p]-\eta)\mathbf{A}_I}\cdot\cos\omega_c(S_n - \tau_n[p] - \eta) \approx e^{S_n\mathbf{A}_I} \tag{14}$$

$$e^{(S_n-\tau_n[p]-\eta)\mathbf{A}_Q}\cdot\sin\omega_c(S_n - \tau_n[p] - \eta) \approx e^{S_n\mathbf{A}_Q}\theta_n[p]. \tag{15}$$

This is because: (**i**) $e^{\hat{\tau}\mathbf{A}_I} \approx \mathbf{I}$ (and similarly for $\mathbf{A}_Q$), since the cavity's baseband dynamics are slow ($Q \gg 1$); and (**ii**) $\sin\omega_c(S_n - \tau_n[p] - \eta) \approx -\theta_n[p]$ and $\cos\omega_c(S_n - \tau_n[p] - \eta) \approx 1$, given our assumptions on $\omega_c$. Similarly,

$$\mathbf{C}_c\bar{\mathbf{x}}_c(pT + T_n) = \mathbf{C}_I\mathbf{x}_I(pT + T_n) + \mathbf{C}_Q\mathbf{x}_Q(pT + T_n), \tag{16}$$

i.e., the voltage at the fixed instants $pT + T_n$ can be evaluated just as well using the I and Q baseband models.

Thus we can replace the cavity-model dynamics of Eq. (11) with (orthogonal) I and Q models. In fact, under Eq. (14) the I model is uncoupled to the bunches altogether and can be neglected in any dynamics analysis (cf. the hueristic arguments for this in [9]). Thus we replace the model of Eq. (11) with

$$\begin{aligned}\mathbf{x}_{\mathrm{Q}}(pT+T_n) &= e^{S_n\mathbf{A}_{\mathrm{Q}}}\mathbf{x}_{\mathrm{Q}}(pT+T_{n-1}) - q_{n-1}e^{(S_n-\tau_{n-1}[p]-\eta)\mathbf{A}_{\mathrm{Q}}}\mathbf{B}_{\mathrm{Q}} \\ &\approx e^{S_n\mathbf{A}_{\mathrm{Q}}}\mathbf{x}_{\mathrm{Q}}(pT+T_{n-1}) - q_{n-1}\theta_{n-1}[p]e^{S_n\mathbf{A}_{\mathrm{Q}}}\mathbf{B}_{\mathrm{Q}}, \quad (17)\end{aligned}$$

and modify Eq. (12) as

$$\begin{aligned}\mathbf{x}_n[p+1] &= \mathbf{A}_{\mathrm{b}}\mathbf{x}_n[p] + \mathbf{B}_{\mathrm{b}}\mathbf{C}_{\mathrm{Q}}e^{(S_n+\tau_n[p]-\eta)\mathbf{A}_{\mathrm{Q}}}\mathbf{x}_{\mathrm{Q}}(pT+T_{n-1}) \\ &\approx \mathbf{A}_{\mathrm{b}}\mathbf{x}_n[p] + \mathbf{B}_{\mathrm{b}}\mathbf{C}_{\mathrm{Q}}e^{S_n\mathbf{A}_{\mathrm{Q}}}\mathbf{x}_{\mathrm{Q}}(pT+T_{n-1}), \quad (18)\end{aligned}$$

thus obtaining a linear model.

As stated, the BTCI model is in fact an $N$-fold switched system (see [10]), but it is also periodic. Thus, to evaluate stability we need only compute the eigenvalues of the single-period system matrix. For convenience define $\mathbf{C}_{\mathrm{b},n} := [-q_n, 0]$ (output matrix for the bunch system), and

$$\begin{aligned}\mathbf{K}_n &:= \mathbf{B}_{\mathrm{Q}}\mathbf{C}_{\mathrm{b},n} \\ \mathbf{H} &:= \mathbf{B}_{\mathrm{b}}\mathbf{C}_{\mathrm{Q}} \\ \mathbf{E}_{n,m} &:= e^{(T_n-T_m)\mathbf{A}_{\mathrm{Q}}}.\end{aligned}$$

Let $\mathbf{x}[p] = [\mathbf{x}_{\mathrm{Q}}^{\mathrm{T}}(pT), \mathbf{x}_1^{\mathrm{T}}[p], \cdots, \mathbf{x}_N[^{\mathrm{T}}p]]$, which behaves according to

$$\mathbf{x}[p+1] = \mathbf{A}\mathbf{x}[p], \quad (19)$$

where

$$\mathbf{A} = \begin{bmatrix} \mathbf{A}_{11} & \mathbf{A}_{12} \\ \mathbf{A}_{21} & \mathbf{A}_{22} \end{bmatrix}, \quad (20)$$

with

$$\begin{aligned}\mathbf{A}_{11} &:= e^{T\mathbf{A}_{\mathrm{Q}}} \\ \mathbf{A}_{12} &:= \begin{bmatrix} e^{(T-T_1)\mathbf{A}_{\mathrm{Q}}}\mathbf{K}_1 & \vdots & \cdots & \vdots & e^{(T-T_N)\mathbf{A}_{\mathrm{Q}}}\mathbf{K}_N \end{bmatrix} \\ \mathbf{A}_{21} &:= \begin{bmatrix} \mathbf{H}e^{T_1\mathbf{A}_{\mathrm{Q}}} \\ \vdots \\ \mathbf{H}e^{T_N\mathbf{A}_{\mathrm{Q}}} \end{bmatrix}\end{aligned}$$

$$\mathbf{A}_{22} := \begin{bmatrix} \mathbf{A}_b & & & & \\ \mathbf{HE}_{2,1}\mathbf{K}_1 & \mathbf{A}_b & & & \\ \mathbf{HE}_{3,1}\mathbf{K}_1 & \mathbf{HE}_{3,2}\mathbf{K}_2 & \mathbf{A}_b & & \\ \vdots & \vdots & \ddots & \ddots & \\ \mathbf{HE}_{N,1}\mathbf{K}_1 & \mathbf{HE}_{N,2}\mathbf{K}_2 & \ldots & \mathbf{HE}_{N,N-1}\mathbf{K}_{N-1} & \mathbf{A}_b \end{bmatrix}.$$

From the $\mathbf{A}$-matrix one can compute the growth rates/stability of the system under any arbitrary bunch spacing $\{T_n\}$ and bunch charges $\{q_n\}$. The normal bunch eigenmodes are governed by $\mathbf{A}_{22}$. Unlike the modal analysis of section 2.1, no phantom bunches need be appended. However, note that for large $N$, say on the order of 100, the system matrix may be ill conditioned. It is also noted that there is one known reference in which a state-space formulation is attempted [11]. However, that model is limited to the trivial case of $N = 1$ and is based effectively on modal analysis.

### 3.1 Control

The form of the controlled BTCI model depends on the bandwidth of the actuator, i.e., the generator. The simplest form is

$$\begin{aligned} \mathbf{x}[p+1] &= \mathbf{A}\mathbf{x}[p] + \mathbf{B}\mathbf{u}[p] \\ \mathbf{y}[p] &= \mathbf{C}\mathbf{x}[p], \end{aligned} \tag{21}$$

where $\mathbf{A}$ has been defined in Eq. (20). Implied in this model is that the control input (i.e., the quadrature modulation of the generator's rf) is stepped no fewer than every $T$ seconds, i.e., with each period. If the generator bandwidth is wide enough and it is desired to update its ouput at even faster rates, then the period-wise autonomous system model of Eq. (19) must be reformulated to account for interperiod state changes (a switched system formulation results). We address the more conservative model of Eq. (21).

Since only the cavity is actuated, the input matrix is given by

$$\mathbf{B} = \begin{bmatrix} \mathbf{B}_Q \\ \mathbf{0} \end{bmatrix}. \tag{22}$$

In practice, the cavity output (voltage) is measurable, as is some bunch phase (and/or energy) information. Depending on the number and bandwidth of the sensors, this may be the average bunch phase or even the bunch phase of all $N$ bunches. In any case, the output matrix is of the form

$$\mathbf{C} = \begin{bmatrix} \mathbf{C}_Q & \mathbf{0} \\ \mathbf{0} & \mathbf{C}_B \end{bmatrix},$$

where $\mathbf{C}_\mathrm{B}$ scales the $2N$ bunch states. If the system is controllable and observable, then an observer can be designed to allow for effective full-state feedback (Ch. 29 of [7]).

## 4 Simulation Results

In order to demonstrate the utility of the new formulation we consider the following $N = 2$-bunch example (inspired by [12]). The system period is $T = 3.68E - 06$, and the period-wise matrices given by

$$\begin{aligned}
\mathbf{A}_\mathrm{Q} &= 1.0\mathrm{E}5 \begin{bmatrix} -5.91 & -1.69 \\ 5.24 & 0 \end{bmatrix} \\
\mathbf{B}_\mathrm{Q} &= \begin{bmatrix} 2.62\mathrm{E}5 \\ 0 \end{bmatrix} \\
\mathbf{C}_\mathrm{Q} &= \begin{bmatrix} 0 & -1.90e5 \end{bmatrix} \\
\mathbf{A}_\mathrm{b} &= \begin{bmatrix} 1.00 & 0.036 \\ -0.047 & 0.998 \end{bmatrix} \\
\mathbf{B}_\mathrm{b} &= \begin{bmatrix} 0 \\ 0.610\mathrm{E}-4 \end{bmatrix} \\
\mathbf{C}_\mathrm{b} &= \begin{bmatrix} 0.122\mathrm{E}-3 & 0 \end{bmatrix}.
\end{aligned}$$

We consider an asymmetric bunch pattern given by $T_1 = 0, T_2 = T/3$, with $q_1 = q_2 = 0.129\mathrm{E} - 5$. The system eigenvalues are

$$\lambda(\mathbf{A}) = \{0.3337 \pm 0.0449i, 0.9991 \pm 0.0403i, 0.9992 \pm 0.0410i\},$$

and since the modulus of one pair is greater than unity, the system is unstable.

It is assumed that the generator can update its output each $T$, but is bandlimited such that it cannot update on a bunch-by-bunch ($T_2$ or $T - T_2$) basis. These assumptions are realistic. The pair $(\mathbf{A}, \mathbf{B})$ is found to be controllable [see Eq. (22)]. We assume that the average phase across the two bunches ($(\tau_1 + \tau_2)/2$) is also measurable. Thus,

$$\mathbf{C} = \begin{bmatrix} \mathbf{C}_\mathrm{Q} & \mathbf{0} \\ \mathbf{0} & \begin{bmatrix} 0.5 & 0 & 0.5 & 0 \end{bmatrix} \end{bmatrix},$$

and the pair $(\mathbf{A}, \mathbf{C})$ is found to be observable, allowing for effective full-state feedback. We choose the desired closed-loop eigenvalues to be

$$\lambda(\mathbf{A} - \mathbf{BK}) = \{0.3400 \pm 0.0400i, 0.9990 \pm 0.0403i, 0.9991 \pm 0.0405i\},$$

whose moduli are less than unity.

Some simulation results are presented in Figure 2. A 1000-V square pulse is applied to the second bunch's second state (energy deviation $\epsilon_2$) for $100T$ seconds. That bunch's phase ($\theta_2$) is open-loop unstable, but is stabilized with our feedback. Note that the frequency-domain-based control design does damp bunch oscillations in the trival $N = 1$ case, but is not realizable in the (symmetric or asymmetric) $N = 2$ case because of our assumed generator update-time contraint, as it requires filtering at frequencies greater than $1/T$.

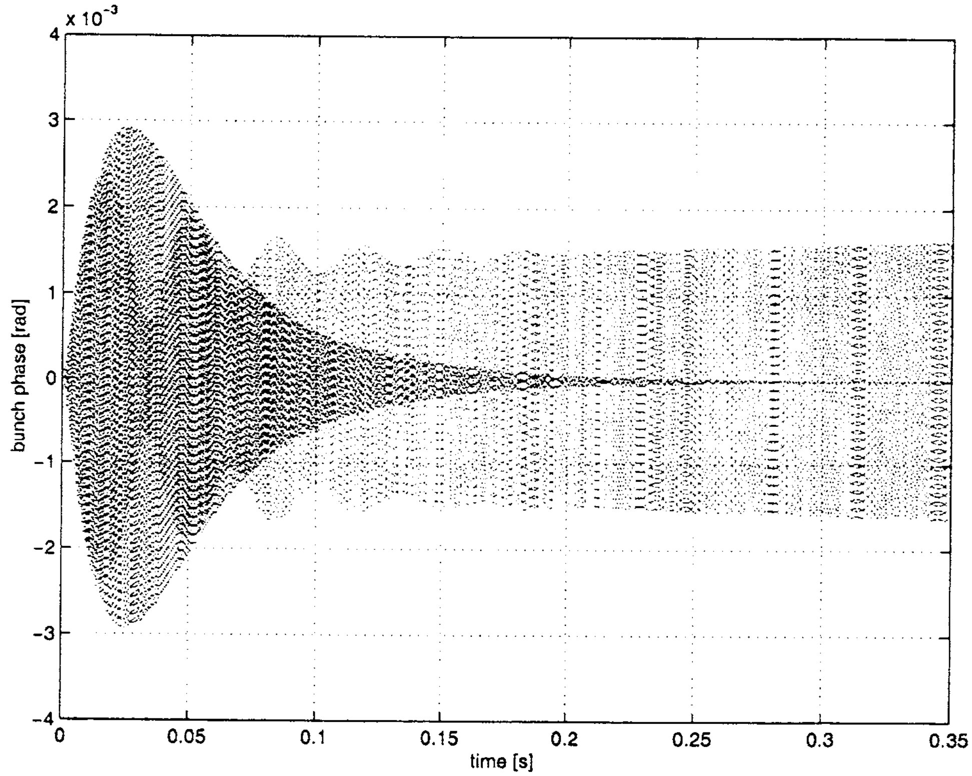


Figure 2: Open-loop (times 10, in green) and closed-loop (in blue) bunch phases.

## 5 Summary and Conclusion

A linear time-domain model has been derived for the bunch-train cavity interaction problem. This general formulation does not require system symmetry and thereby leads to a model with no phantom states. The model is currently being utilized to predict stability for arbitrary bunch spacings in the Advanced Photon Source at Argonne National Laboratory (Illinois). The final version of this paper will report on some of these results.

The BTCI control problem motivated this work: state-space and time-domain methods have not been developed for this problem, and this lack is seen in the ad-hoc manner in which practical control systems are designed. An example is the paper [1], in which many loops are nested about the plant. The new formulation presented herein answers this need. It is also applicable to general switched, quasiperiodic systems that undergo jitter.

## 6 Acknowledgments

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## References

[1] F. Pedersen. RF cavity feedback. In *B Factories: The State-of-the Art in Accelerators, Detectors, and Physics*, 1992.

[2] D. Teytelman et al. Architecture and technology of 500 msample/s feedback systems for control of coupled-bunch instabilities. In *Proceedings of the 7th International Conference on Accelerator and Large Experimental Physics Control Systems*, 1999.

[3] H. S. Kang et al. Cure of coupled bunch instabilities in PLS storage ring. In *Proceedings of the 2000 European Particle Accelerator Conference*, 2000.

[4] P. B. Wilson. Fundamental-mode RF design in e+ e- storage ring factories. In *Frontiers of Particle Beams: Factories with e+ e- Rings*, http://www.slac.stanford.edu/pubs/slacpubs/6000/slac-pub-6062.html, 1992.

[5] S. Prabhakar. *New Diagnostics and Cure for Couple-Bunch Instabilities.* PhD thesis, Stanford University, February 1999. http://www.slac.stanford.edu/pubs/slacreports/slac-r-554.html.

[6] A. Papoulis. Error analysis in sampling theory. *Proceedings of the IEEE*, 54(7), 1966.

[7] W. J. Rugh. *Linear System Theory.* Prentice-Hall, second edition, 1996.

[8] A. Papoulis. *Signal Analysis.* McGraw–Hill, 1977.

[9] S.Y. Zhang and W.T. Weng. A new formulation of longitudinal coherent instabilities. In *Proceedings of the 1993 Particle Accelerator Conference*, 1993.

[10] M. S. Branicky et al. A unified framework for hybrid control: Background, model, and theory. In *Proceedings of the 33rd IEEE Control and Decision Conference*, 1994.

[11] D. Boussard and E. Onillon. Applications of the methods of optimum control theory to the RF system of a circular accelerator. Technical Report SL/93-09/RFS, CERN, 1993.

[12] Y. Chao et al. Damping the $\pi$-mode instability in the SLC damping rings with a passive cavity. In *Proceedings of the Fifteenth International Conference on High Energy Accelerators*, 1992.